Stress transmission in granular matter

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Abstract
The transmission of forces through a disordered granular system is studied by
means of a geometrical–topological approach that reduces the granular packing
into a set of layers. This layered structure constitutes the skeleton through which
the force chains set up. Given the granular packing, and the region where the
force is applied, such a skeleton is uniquely defined. Within this framework, we
write an equation for the transmission of the vertical forces that can be solved
recursively layer by layer. We find that a special class of analytical solutions
for this equation are Lévi-stable distributions. We discuss the link between
criticality and fragility and we show how the disordered packing naturally
induces the formation of force chains and arches. We point out that critical
regimes, with power law distributions, are associated with the roughness of
the topological layers, whereas fragility is associated with local changes in the
force network induced by local granular rearrangements or by changes in the
applied force. The results are compared with recent experimental observations
in particulate matter and with computer simulations.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In recent years many experiments [1–9], simulations [10–12] and theoretical
approaches [13–17] have been devoted to the study of propagation of forces through gran-
ular aggregates. One of the interesting features that arise from these studies is the formation of
strong inhomogeneities in the spatial distribution of contact forces (formation of force chains),
where a small fraction of the grains carries most of the total force (see figure 1). These chains
typically extend on space scales much larger than the grain dimensions and are associated
with broad stress distributions [3, 4, 6–8, 10, 11]. Moreover, simulations and experiments show
strong rearrangements of the force chains under small changes in the compression axes, sug-
gesting a 'fragile' behaviour associated with the force-chain structure [11, 16, 17]. In this paper
we show that the formation of force chains, the broadening of the force distribution and the

Figure 1. Force chains through a disordered granular packing made by a mixture of photoelastic discs with two different sizes. The luminosity is proportional to the amount of stress carried (see footnote 3 below).

'fragile' behaviour are direct consequences of the topological disorder in the granular packing structure.

In order to study the static stress distribution, we use a scalar model where the force balance is considered in the vertical direction only and torques are neglected. This approach is similar to several theoretical works already proposed in the literature [4, 11, 13, 14, 16] and in particular to the well known q model [4, 13]. In our model, however, we explicitly consider a *disordered* arrangement of grains (instead of placing the grains on a regular array and introducing randomness in the transmission term, as for the q model). This leads to strong differences and some important, novel consequences (see also the appendix).

2. Topological characterization of the granular packing

The complete description of a granular structure in two and three dimensions requires in general the information about the positions, sizes and shapes of all the grains, but to describe the propagation of the weight (vertical forces) one can use only the topological information about the neighbours and the contacts among grains combined with the knowledge about the direction and intensity of the transmission among neighbouring grains (see figure 2).

To univocally define the neighbouring structure we use the 'navigation map' [18], which consists in the construction of a space-filling cellular structure with generalized Voronoï [19] cells, where the faces are made by the set of points equidistant to the surfaces of the grains. From this construction two grains are first neighbours if they share a common face. The neighbouring structure is topologically defined by the matrix C:
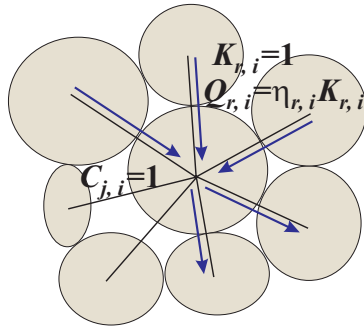
Figure 2. A granular packing is topologically described by the neighbouring matrix $C_{i,j}$ (lines). The transmission of the weight is through adjacent grains and this structure is the Dodds network, which is defined by the 'kissing' matrix $K_{i,j}$ (arrows). The force propagates in one direction and inhomogeneously among adjacent neighbours. This can be described by the weight transmission matrix $Q_{i,j} = \eta_{i,j} K_{i,j}$.

$$C_{i,j} = \begin{cases} 1 & \text{if grain } i \text{ is first neighbour of grain } j \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

which is symmetric with zeros on the diagonal. This matrix describes the neighbouring structure of the packing. However, by construction, two neighbouring grains i, j (with $C_{i,j} = 1$) might not be in contact, whereas two grains which are in contact are surely neighbours. Since the force propagates only through grains which are in contact, it follows that to study the force propagation one must also take into account the topological structure of the inter-grain contacts (Dodds network) [20]. This can be done by defining the 'kissing' matrix K:

$$K_{i,j} = \begin{cases} 1 & \text{if grain } i \text{ is in contact with grain } j \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(with $K_{i,j} = K_{j,i}$, and $K_{i,i} = 0$). The number of grains which are in contact ('kiss') with a given grain i is

$$n_i = \sum_{j=1}^{N} K_{i,j} \tag{3}$$

where N is the total number of grains in the system. An upper bound on the number of adjacent grains n_i arises from geometrical considerations: it is indeed known that no more that six equal discs and no more than 12 equal spheres can stay in touch with a central one (kissing numbers) [21, 22]. On the other hand, when the sizes can fluctuate and the grains can assume different shapes, these numbers might change dramatically. However, for grain sizes distributed in a finite range, one can always find an upper bound on the number of neighbours in contact with a given grain. For instance, when the grain sizes are homogeneous and the shapes isotropic, these upper bounds are very close to the above values of six and 12, in two and three dimensions respectively. A lower bound on n_i is given by the mechanical stability condition, as we shall see shortly.

3. Weight propagation

We study the propagation of the vertical component of the forces (weights) from grain to grain. This is a conservative system: the total weight on the bottom of the stack is equal to the

vertical force applied on the top plus the proper weights of the grains in the stack (we neglect the presence of walls, that we put at infinity).

We describe the propagation of the vertical component of the forces inside the granular structure in terms of the following equation:

$$w_i = w_i^0 + \sum_{j=1}^{N} w_j Q_{j,i} \tag{4}$$

with w_i being the vertical force applied at the barycentre of grain i. The quantities $Q_{j,i}$ are the elements of the *weight transmission matrix* Q; they are associated with the kissing matrix through the relation $Q_{j,i} = \eta_{j,i} K_{j,i}$, where the quantities $\eta_{j,i}$ take values between zero and unity. These quantities are asymmetric: if $\eta_{j,i} > 0$ then $\eta_{i,j} = 0$ (but not vice versa), giving in this way the orientation of the force network.

The symbol w_i^0 indicates the proper weight of grain i with an exception for the top layer of the granular stack where one can apply an external force. In this case, the symbol w_i^0 indicates the proper weight of grain i plus the portion of the external force acting on this grain.

Equation (4) is a rather general expression: it assumes only that—as it is—the vertical force propagates only through grains which are in contact. Equation (4) can also take into account nonlinear effects if one supposes that the quantities $Q_{j,i}$ are dependent on the force w_i. However, in equation (4) the vectorial nature of the forces and the tensorial nature of the stresses have been completely neglected. Nevertheless, we shall show that the scalar description given by equation (4) is very powerful and can take into account complex phenomena such as the formation of force chains and arches or the fragility.

In equation (4) the whole information about the granular packing is contained in the weight transmission matrix Q, which must therefore be consistent with a real connection network in a physically realizable granular packing. This gives some constraints on Q. The conservation of the total weight implies that the sum over each column of the matrix Q must be equal to unity, i.e.

$$\sum_{j=1}^{N} Q_{j,i} = 1. \tag{5}$$

The stability under gravity imposes that each grain must lie on at least three other grains in three dimensions or two grains in two dimensions, therefore

$$s_i = \sum_{j=1}^{N} \theta(Q_{j,i}) \geqslant D \tag{6}$$

with D the dimensionality of the space and $\theta(x)$ the step function ($\theta(x) = 1$ when $x > 0$; $\theta(x) = 0$ when $x \leqslant 0$).

The grain i receives the vertical force from a number

$$m_i = \sum_{j=1}^{N} \theta(Q_{j,i}) \tag{7}$$

of neighbours, and shares its total weight w_i through

$$s_i = \sum_{j=1}^{N} \theta(Q_{i,j}) \tag{8}$$

neighbours. Clearly, $m_i + s_i = n_i$ is the number of grains adjacent to i and, from the stability condition (equation (6)), we must have $n_i \geqslant D$, giving therefore a lower bound on this quantity.

4. A linear equation

We now seek for solutions of equation (4). Let us first consider the special case where the elements $Q_{i,j}$ of the weight transmission matrix depend on the granular packing structure only (i.e. the $Q_{i,j}$ do not depend on w_j). In this case equation (4) is a linear equation, and its solution can be written straightforwardly. Indeed, an equivalent way to write equation (4) is

$$\sum_{j=1}^{N} w_j(\delta_{j,i} - Q_{j,i}) = w_i^0 \tag{9}$$

with $\delta_{i,j}$ the Kronecker delta. When the $Q_{j,i}$ are independent of w_j, equation (9) is a set of N linear equations with N unknowns. If the determinant of $I - Q$ is different from zero, this system will admit a unique solution. Physically this condition corresponds to imposing that in the force network there are no closed oriented rings. In terms of the elements of the force transmission matrix this implies that, for any given i, and for any arbitrarily chosen set of site indices $\{j_1, \ldots, j_k\}$, it must hold that

$$Q_{i,j_1} Q_{j_1,j_2} Q_{j_2,j_3} \ldots Q_{j_k,i} = 0. \tag{10}$$

Note that this condition implies $Q^x = 0$ for any $x > N$. Indeed it is impossible to make a non-intersecting path with a length larger than the number of grains in the system.

Equation (9) can be also written in a vectorial notation

$$w(I - Q) = w^0 \tag{11}$$

and, when the determinant of $I - Q$ is different from zero, its solution is

$$w = w^0(I - Q)^{-1}. \tag{12}$$

Although equation (12) is formally trivial, its solution cannot be easily pursued analytically in the general case and it might become numerically untreatable for large samples. Therefore, in the following we develop an alternative approach, which considers the system as structured into a set of layers and makes it possible to solve equation (4) recursively.

5. Construction of the layered structure

Any granular packing can be formally reduced into a stacking of layers where the vertical force components are transmitted downward from layer to layer. The elements of these layers can be single grains or sets of grains that collectively behave as a single element that receives the weight from the layer above and transmits it to the layer below.

In order to construct such a layered system, let us first label all the grains at the top of the stacking (on which—eventually—the external load is applied) as belonging to the layer zero ($\mathcal{L}_0$). Let us also consider as belonging to $\mathcal{L}_0$ all the grains which transmit some weight to grains already labelled as belonging to $\mathcal{L}_0$. Once $\mathcal{L}_0$ is defined, the first layer ($\mathcal{L}_1$) is made by the set of grains which are neighbours of grains in $\mathcal{L}_0$ and do not belong to $\mathcal{L}_0$. Moreover, all the grains which are not yet classified in $\mathcal{L}_0$ and that transmit some force to grains already classified in $\mathcal{L}_1$ belong to $\mathcal{L}_1$. Analogously, in general, the layer $\mathcal{L}_\alpha$ is constituted of grains (i) that are neighbours of grains (j) in layer $\mathcal{L}_{\alpha-1}$ and do not belong to any already classified layer ($i \in \mathcal{L}_\alpha$ if $i \notin \mathcal{L}_\beta$ with $\beta < \alpha$, and if there exists at least one $j \in \mathcal{L}_{\alpha-1}$ such that $C_{i,j} = 1$). In addition, all the non-classified grains which transmit some force to grains which have been already classified in $\mathcal{L}_\alpha$ belong to $\mathcal{L}_\alpha$ ($i \in \mathcal{L}_\alpha$ if $i \notin \mathcal{L}_\beta$ with $\beta < \alpha$, and if there exists at least one $j \in \mathcal{L}_\alpha$ such that $\eta_{i,j} > 0$).
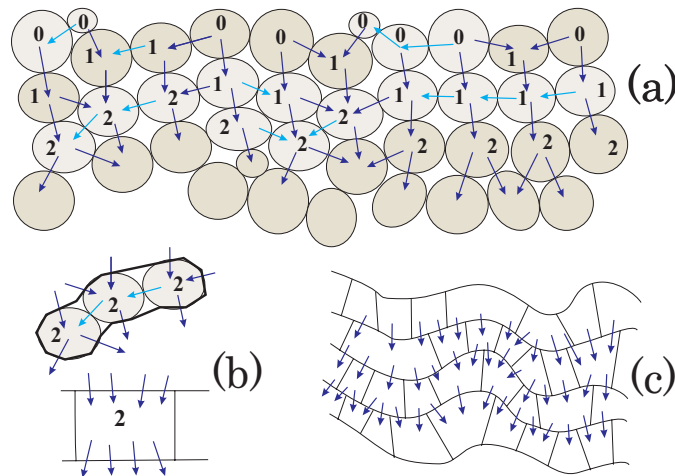
Figure 3. (a) The layer $\mathcal{L}_0$ is made by the set of grains on which the force is applied plus all the grains that transmit force to them. $\mathcal{L}_\alpha$ is made of the grains which are neighbours to $\mathcal{L}_{\alpha-1}$ plus the grains which transmit some force to the grains in $\mathcal{L}_\alpha$ but have not yet been classified. (b) Clusters of adjacent grains in the same layer which transmit the force among themselves are considered as single 'elements'. (c) The resulting structure is an SSI [23] layered packing made of elements which receive some weight from the layer above and transmit it to elements in the layer below.

In the system of layers that we have now constructed we have two kinds of grain: (i) some grains receive the weight from the layer above only, and transmit it only to the layer below; (ii) other grains receive or transmit weight to grains in the same layer. We want to reduce the packing to a set of 'elements' where every element k (which belongs to some layer, $k \in \mathcal{L}_\alpha$) receives the weight from elements (j) in the layer above ($j \in \mathcal{L}_{\alpha-1}$) and transmits the weight to elements j' in the layer below ($j' \in \mathcal{L}_{\alpha+1}$). To this purpose we identify as single *elements* of the layered structure the grains of type (i), and the clusters of grains of type (ii).

A schematic graphical representation of the procedure to identify the granular layers and their elements is given in figure 3.

6. Recursive solutions in layered packings

In the layered system that we have constructed in the previous section the weight is transmitted layer by layer from the top of the stacking (layer $\mathcal{L}_0$, which receives the external load) down to the bottom layer $\mathcal{L}_h$ (layer at distance h from the top one).

In analogy with equation (4), it follows that the weight on a given element in layer $\mathcal{L}_\alpha$ ($\alpha \geqslant 1$) is given in terms of the weights of the elements on the layer above $\mathcal{L}_{\alpha-1}$ through the following equation:

$$w_i = w_i^0 + \sum_{j \in \mathcal{L}_{\alpha-1}} w_j Q_{j,i} \qquad \text{with} \quad i \in \mathcal{L}_\alpha. \tag{13}$$

Note that in this case the indices i and j refer to *elements* of the packing instead of *grains*. The definition of $Q_{i,j}$ is identical to that given before, but it refers to a granular structure which has been reduced to a system of elements disposed into layers which make an SSI network [23] (see figure 3(c)). The forces on the top layer are known: $w_i = w_i^0$ (for $i \in \mathcal{L}_0$). Therefore, by using recursively equation (13), one can calculate step by step the forces w_i from the first layer to the bottom. This recursive strategy can be applied to solve equation (13) even in the
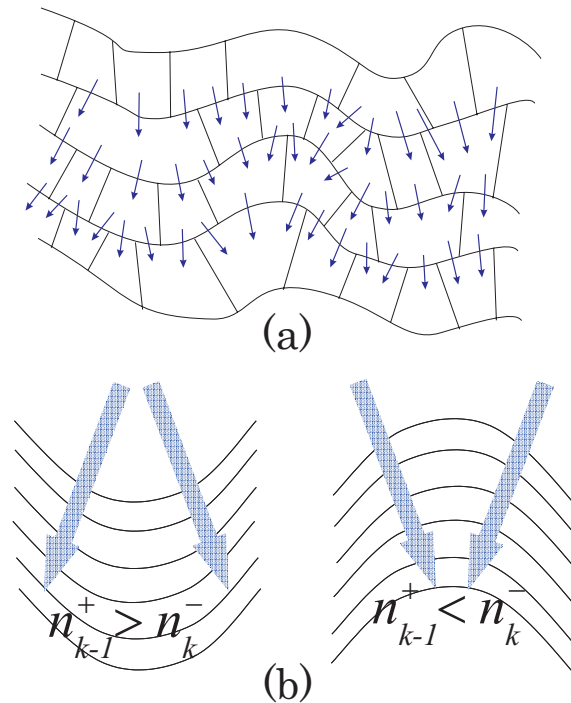
Figure 4. (a) The layers in disordered granular packings are made of sets of elements placed on rough surfaces (wiggled lines in two dimensions) that make a dimpled landscape rich in 'valleys' and 'mountains'. For geometrical reasons, successive layers have similar local curvature and the roughening is persistent, up to a certain distance, through the layers. (b) The force is concentrated (or 'focalized') when transmitted through 'mountains' whereas it is diffused ('defocalized') in the 'valley' region.

nonlinear case, with the only assumption that the quantities $Q_{j,i}$ must depend only on the forces applied from the layers above.

7. Force chains, arches and fragility

In experiments and computer simulations of force propagation through granular matter an important fact is observed: only a small proportion of the grains carries most of the weight. The force propagates through 'chains', and these chains generate 'arches'. The formation of these chains implies that there are some regions of the packing where the weight tends to *concentrate in*, and there are other regions where the weight tends to *diffuse out*.

In order to understand this mechanism in term of our layered structure, let us consider the shape of these layers. In a crystalline-ordered packing (an FCC stacking of balls, for instance) the layers are made of sets of elements disposed on flat surfaces which are parallel to each other (or straight lines in two-dimensional packings). When disorder is introduced, these surfaces start to bend and become rough. The global curvature will remain zero, but locally 'valleys' and 'mountains' will make a dimpled landscape. These rough surfaces are stacked one upon the other and therefore the roughening of neighbouring layers must be *similar* and the local curvature is expected to be preserved for a few layers (see figure 4).

Any given element of the layered packing receives the weight from the elements in the layer above and distributes it among the neighbours in the layer below. Because of the

disorder, some elements have more neighbours in the layer above than others. Clearly, if we have no structural correlation among the layers, we expect that the elements that receive the weight from more elements will end up—on average—with a larger weight than the others with fewer neighbours above. This mechanism would be amplified layer after layer if this property of abundance/deficiency of neighbours locally propagated through the layers. This is indeed the case in granular packings, where, for geometrical reasons, the roughening of the layers must be persistent through a certain number of layers. The persistence length is strictly related to the average chain length and gives the measure of the amplification effect. In particular, we have that elements in 'valleys' tend to dissipate the force (*defocalizing effect*) by propagating it to a larger number of neighbours with respect to the number from which they receive it, whereas elements in the 'mountains' tend to concentrate the weight (*focalizing effect*) by transmitting the force to a smaller number of elements with respect to the number from which they receive it (see figure 4). The formation of arches in an 'hands-on' simple experiment[3], made with photoelastic material between two polarizing filters, is shown in figure 1. The result of a computer simulation of weight propagation through an SSI structure is shown in[4] figure 5. As one can see, most of the stress propagates through *force chains* which make *arches* by self-interconnecting into an intricate network.

The readjustment of a few grains or a change in the point or direction of the applied force strongly affects the layered structure. Indeed, a local change modifies locally the composition and curvature of the layer. This modification propagates through the whole system by changing the roughening and the set of grains in the layers. The final effect is a modification in the whole structure of the force-chain network. This long-range propagation, associated with the focalizing/defocalizing effects, is responsible for some of the critical effects in granular matter and it is at the origin of the fragility in these systems.

8. Force distribution: an analytical approach

In this section we calculate analytically, in the linear case, the probability distribution for the vertical force components. The probability $P(w, r)$ of finding a force w at the site r is related to the probability that the grains at sites j have weights w_j, and that the propagation of these weights through the contact matrix $Q_{j,r}$ produces exactly w, which is formally written (from equation (4))

$$P(w, r) = \prod_{k,j} \int_0^1 \mathrm{d}Q_{k,r} \Omega(\mathbf{Q}) \int \mathrm{d}w_j P(w_j, j) \delta\left(w - w_r^0 - \sum_{k'} w_{k'} Q_{k',r}\right) \tag{14}$$

where $\delta(x)$ is the Dirac delta function and the quantity $\Omega(\mathbf{Q})$ is the probability of a given force transmission matrix $\mathbf{Q}$.

Let us introduce the Fourier transform of $P(w, r)$ and its inverse

$$\begin{aligned} \hat{P}(\varphi, j) &= \mathcal{N} \int \mathrm{d}w \mathrm{e}^{\mathrm{i}w\varphi} P(w, j) \\ P(w, j) &= \int \mathrm{d}\varphi \mathrm{e}^{-\mathrm{i}w\varphi} \hat{P}(\varphi, j) \end{aligned} \tag{15}$$

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[3] This was a prototypal exhibit, T Aste, for the exhibition *Simple and Complex (1999)*.
[4] These simulations will be presented in more detail in a forthcoming paper.
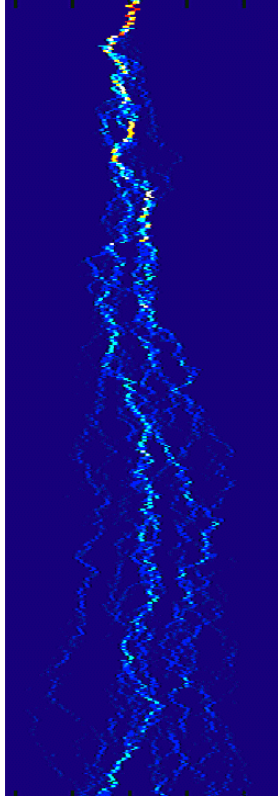
Figure 5. Force chains resulting from a computer simulation of weight transmission through an SSI granular system.

with $\mathcal{N}$ a constant which takes into account the normalization to unity. By using this transformation, equation (14) becomes

$$P(w, r) = \mathcal{N} \prod_{kj} \int_0^1 \mathrm{d}Q_{k,r} \Omega(Q) \int \mathrm{d}w_j \int \mathrm{d}\varphi_j \mathrm{e}^{-\mathrm{i}w_j \varphi_j} \hat{P}(\varphi_j, j) \int \mathrm{d}\phi \mathrm{e}^{-\mathrm{i}(w-w_r^0 - \sum_{k'} w_{k'} Q_{k',r})\phi}$$

(16)

where the Dirac delta function has been written as

$$\delta(x) = \mathcal{N} \int \mathrm{d}\phi\, \mathrm{e}^{-\mathrm{i}x\phi}.$$

(17)

Equation (16) can be re-written as

$$P(w, r) = \mathcal{N} \int \mathrm{d}\phi\, \mathrm{e}^{-\mathrm{i}(w-w_r^0)\phi} \prod_{k \in \mathcal{I}_r} \int_0^1 \mathrm{d}Q_{k,r} \Omega(Q) \prod_{j \in \mathcal{I}_r} \int \mathrm{d}\varphi_j \hat{P}(\varphi_j, j) \int \mathrm{d}w_j \mathrm{e}^{-\mathrm{i}w_j \varphi_j} \mathrm{e}^{\mathrm{i}w_j Q_{j,r}\phi}$$

(18)

where the integral over the Q is restricted only to the set $\mathcal{I}_r$ of elements which are adjacent to the element r and are transmitting the force to this element (i.e. $Q_{j,r} \neq 0$). The integration over w_j yields

$$P(w, r) = \mathcal{N} \int \mathrm{d}\phi\, \mathrm{e}^{-\mathrm{i}(w - w_r^0)\phi} \prod_k \frac{1}{\mathcal{N}} \int_0^1 \mathrm{d}Q_{k,r}\, \Omega(Q) \prod_{j \in \mathcal{I}_r} \hat{P}(Q_{j,r}\phi, j) \quad (19)$$

from equation (15), its Fourier transform is

$$\hat{P}(\psi, r) = \mathcal{N}^2 \int \mathrm{d}\phi \int \mathrm{d}w\, \mathrm{e}^{\mathrm{i}w\psi} \mathrm{e}^{-\mathrm{i}(w - w_r^0)\phi} \prod_k \frac{1}{\mathcal{N}} \int_0^1 \mathrm{d}Q_{k,r}\, \Omega(Q) \prod_{j \in \mathcal{I}_r} \hat{P}(Q_{j,r}\phi, j). \quad (20)$$

The integration over w gives finally

$$\hat{P}(\psi, r) = \mathcal{N}\mathrm{e}^{\mathrm{i}w_r^0\psi} \prod_k \frac{1}{\mathcal{N}} \int_0^1 \mathrm{d}Q_{k,r}\, \Omega(Q) \prod_{j \in \mathcal{I}_r} \hat{P}(Q_{j,r}\psi, j). \quad (21)$$

If we assume that $\Omega(Q)$ factorizes,

$$\Omega(\{Q\}) = \prod_{j,k} \tilde{\Omega}_j(Q_{j,k}) \quad (22)$$

equation (21) reduces to

$$\hat{P}(\psi, r) = \mathcal{N}\mathrm{e}^{\mathrm{i}w_r^0\psi} \prod_{j \in \mathcal{I}_r} \frac{1}{\mathcal{N}} \int_0^1 \mathrm{d}\zeta_j\, \tilde{\Omega}_j(\zeta_j)\hat{P}(\zeta_j\psi, j). \quad (23)$$

9. Lévi-stable force distributions in a layered-structured system

When the force propagates downward in a layered system, an element in a layer α will receive the weight only from a set of elements in the layer above $(\alpha - 1)$ and will transmit it to a set of elements in the layer below $(\alpha + 1)$. Let us suppose that the probability distribution of the force depends only on the layer number α. This allows us to substitute into equation (23) the quantity $\hat{P}(\psi, r)$ with $\hat{P}_\alpha(\psi)$

$$\hat{P}_\alpha(\psi) = \mathcal{N}\mathrm{e}^{\mathrm{i}w_\alpha^0\psi} \prod_{j \in \mathcal{I}_r} \frac{1}{\mathcal{N}} \int_0^1 \mathrm{d}\zeta_j\, \tilde{\Omega}_j(\zeta_j)\hat{P}_{\alpha-1}(\zeta_j\psi) \quad (24)$$

where w_α^0 is the proper weight of the elements in layer α.

Supposing that the load on a given element in the layer $\alpha - 1$ is uniformly distributed among the $n_{\alpha-1}^+$ neighbours in the layer α, the term $\tilde{\Omega}_j(\zeta_j)$ becomes $\tilde{\Omega}_j(\zeta_j) = \delta(\zeta_j - 1/n_{\alpha-1}^+)$. This simplifies equation (24) to

$$\hat{P}_\alpha(\psi) = \mathcal{N}\mathrm{e}^{\mathrm{i}w_\alpha^0\psi} \left[\frac{1}{\mathcal{N}} \hat{P}_{\alpha-1} \left(\frac{\psi}{n_{\alpha-1}^+} \right) \right]^{n_\alpha^-} \quad (25)$$

where n_α^- is the number of grains in layer $\alpha - 1$ from which an element in layer α receives the force. One can directly verify that equation (25) is satisfied by choosing $\hat{P}_\alpha(\psi)$ in the form

$$\hat{P}_\alpha(\psi) = \exp\left(\mathrm{i} \sum_{\lambda=0}^\alpha \frac{c_\lambda}{c_\alpha} w_\lambda^0 \psi \right) \exp\left(-\mu\sigma_\alpha |\psi|^\gamma + \mathrm{i}\beta\sigma_\alpha \tan\left(\frac{\pi\gamma}{2} \right) \psi|\psi|^{\gamma-1} \right) \quad (26)$$

with μ and γ arbitrary numbers and $\beta = 0$ for $\gamma = 1$ or β arbitrary for $\gamma \neq 1$. The other parameters in equation (26) are

$$\sigma_\alpha = \prod_{\nu=1}^\alpha n_\nu^- (n_{\nu-1}^+)^{-\gamma} \quad (27)$$

and

$$c_\alpha = \prod_{\nu=1}^\alpha \frac{n_{\nu-1}^+}{n_\nu^-}. \quad (28)$$

Expression (26) might seem complicated, but it is a broadly studied expression: it is the Fourier transform of a symmetric Lévy–Kintchine stable distribution [24]. It describes a probability distribution which is peaked around the value $\sum_{\lambda=0}^{\alpha} c_\lambda w_\lambda^0 / c_\alpha$. It has a width proportional to $\sigma^{1/\gamma}$, and it is asymmetric with a skewness proportional to $\beta \in [-1, 1]$ (β positive distribution skewed to the right, β negative distribution skewed to the left). The parameter $\gamma \in (0, 2]$ is called the characteristic exponent. For instance, when $\gamma = 2$ the distribution is a Gaussian, whereas for $\gamma = 1$ and $\beta = 0$ it is a Lorentzian. These distributions are stable with respect to the sum of a set of stochastic variables and therefore appear naturally in the context of the central-limit theorem. For $\gamma < 2$ they have tails which decrease as power laws with exponents that tend to $-(\gamma + 1)$ for very large deviations. In general the moment of order n becomes infinite when $n > \gamma$; it turns out therefore that the variance is only defined when $\gamma = 2$, whereas the average is defined for $\gamma > 1$.

From equations (26)–(28) two opposite scenarios emerge naturally depending on the values of the parameters n^+ and n^- through the layered structure. Let us first consider the case where $n_\nu^- > n_{\nu-1}^+$ (the elements receive the weight from a number of elements which is *larger* than the number of elements to which the force is transmitted). From equations (27) and (28) it follows that in this case the sum $\sum_{\lambda=0}^{\alpha} c_\lambda w_\lambda^0 / c_\alpha$ tends to diverge with α and consequently the value of σ tends to infinity. This is the mathematical description of the *formation of a force chain* where, layer after layer, the force in the chain increases (*focalization effect*) with an associated broadening in the force distribution which becomes a power law with average and spread that tend to infinity. The opposite scenario corresponds to the case when $n_\nu^- < n_{\nu-1}^+$ (the elements receive the weight from a number of elements which is *smaller* than the number of elements to which the force is transmitted). In this case, equations (27) and (28) yield a force distribution which has $\sigma \to 0$ and concentrates around the proper weight of the elements. The elements receive zero weight from above and diffuse their weights outward from this region (*defocalizing effect*). These are the uncharged grains that stay in the regions below the arches.

10. Conclusions

We have presented a study of the weight propagation through a granular stacking which is based on a realistic geometrical–topological description of the packing structure. The guiding idea was to analyse the disordered packing as structured in *layers* made of *elements* which receive the weight from the layer above and transmit it to the layer below. This system is an SSI packing and the equation for the vertical forces can be solved recursively. We show that the appearance of inhomogeneous force distributions into networks of *force chains* and *arches* are naturally found as a geometrical consequence of the rough shape of the layers in disordered packings. The force is *focalized* into chains or diffused out and *defocalized* depending on the local curvature of the layered structure. Strong changes in the force-network structure can be generated by small rearrangements in the packing which are amplified by the geometrical correlations through the layered system. This leads to an intrinsic fragile behaviour of these systems. We find a class of solutions for the distribution of forces which falls in the class of Lévy–Kintchine stable distributions. The focalizing/defocalizing effects are analytically retrieved for these solutions and associated with the local topological properties (n^+, n^-) of the layered structure.

Acknowledgment

Appendix. Special cases: the q-model and lattice-based models

In the literature several models have been presented for the force propagation in granular matter. Most of these models assume that the granular packing is stacked in an ordered lattice-like structure and the disorder is introduced by supposing an inhomogeneous propagation of the force among neighbouring grains. This is—for instance—the working framework of the well known q-model [4, 13]. It might be important to point out that equation (4) becomes identical to that of the q-model if

- the equation is linear ($Q_{i,j}$ independent of w_j),
- the granular packing is structured in layers,
- the force propagates only downward from layer to layer and
- the constraint given by equation (6) is satisfied with the equality.

Therefore, the approach presented in this paper is applicable to the q-model, but not vice versa. In particular, in the framework of the q-model, the layer roughening (which is the mechanism that leads us to the formation of force chains and arches) is associated with the probability that some neighbouring grains transmit no force, whereas the geometrical correlations between successive layers must be taken into account by imposing some 'propagation' of these broken bonds. These are indeed the mechanisms, proposed in the literature, which allow us, within the framework of a lattice-based model (such as the q-model), to obtain the formation of arches and critical force distributions [14]. In our approach these features arise directly from the geometrical nature of the layer system.

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